

CONSTANT MEAN CURVATURE SURFACES IN $\mathbb{M}^2(c) \times \mathbb{R}$ AND FINITE TOTAL CURVATURE

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ABSTRACT. We consider surfaces with parallel mean curvature vector field and finite total curvature in product spaces of type $\mathbb{M}^n(c) \times \mathbb{R}$, where $\mathbb{M}^n(c)$ is a space form, and characterize certain of these surface. When $n = 2$, our results are similar to those obtained in [5] for surfaces with constant mean curvature in space forms.

1. INTRODUCTION

A classical result in the theory of constant mean curvature surfaces (cmc surfaces), proved by H. Hopf [17], says that round spheres are the only cmc spheres in Euclidean space. Hopf's result was extended, by S.-S. Chern [11], to spheres immersed in 3-dimensional space forms. The key ingredient in both papers [17] and [11] is the fact that there exists a quadratic differential form on the surface which is holomorphic when the mean curvature is constant. This form is given by the $(2, 0)$ -part of the quadratic form \mathcal{Q} defined on a cmc surface Σ in a space form by

$$\mathcal{Q}(X, Y) = \langle AX, Y \rangle,$$

for any vector fields X and Y tangent to Σ , where A is the self adjoint operator associated to the second fundamental form of Σ . We note that, if $A_0 = A - HI$ is the traceless part of A , where H is the mean curvature of the surface Σ , then it is easy to see that A_0 vanishes, which means that Σ is an umbilical surface, if and only if the $(2, 0)$ -part of \mathcal{Q} vanishes.

Since the traceless second fundamental form of a surface measures how much it deviates from being totally umbilical, it is natural to study the geometry and topology of complete cmc surfaces with finite total curvature, in the sense that the integral of $|A_0|^2$ is finite. For instance, P. Bérard, M. do Carmo, and W. Santos [5] proved that a cmc surface in a space form $M^3(c)$, $c \leq 0$ with $H^2 > c$ and finite total curvature must be compact. This result was extend to the case of higher codimension in [8] and in [7]. On the other hand, Ph. Castillon [10] considered cmc hypersurfaces in hyperbolic spaces \mathbb{H}^{n+1} with $H^2 < 1$ and finite total curvature. He showed, among others, that such hypersurfaces are diffeomorphic to the interior of a compact manifold. The geometry of cmc surfaces in space forms is also related to the finiteness of the index of the Jacobi operator as described in the works of A. da Silveira [18] and P. Bérard, M. do Carmo, and W. Santos [6].

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The next step in the study of cmc surfaces was to consider isometric immersion on product spaces of type $M^2(c) \times \mathbb{R}$, where $M^2(c)$ is a space form of constant curvature c . In this case, U. Abresch and H. Rosenberg, in their celebrated paper [1], introduced a differential form (the Abresch-Rosenberg differential) which is holomorphic on cmc surfaces and determined those surfaces on which this holomorphic differential vanishes.

Taking into the account the relation between the quadratic form \mathcal{Q} and the operator A_0 , it seems natural to ask if an operator similar to A_0 can be used to study the geometry of cmc surfaces in product spaces

In our paper we give an affirmative answer to this question and we actually obtain some more general results, for surfaces with parallel mean curvature vector field (pmc surfaces) immersed in spaces of type $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a space form. Thus, we use the operator S studied in [3] and [12] to characterize certain pmc surfaces with *finite total curvature*, i.e., such that the integral of $|S|^2$ is finite. We pay a special attention to cmc surfaces in $M^2(c) \times \mathbb{R}$ and obtain similar results to those in [5] for cmc surfaces in space forms.

As the main result, we prove that, if the norm of the second fundamental form of a complete non-minimal pmc surface Σ in $M^n(c) \times \mathbb{R}$ is bounded and the surface has finite total curvature then the function $|S|$ goes to zero uniformly at infinity (see Theorem 3.1 and Corollary 3.3). As applications, we prove lower bounds estimates for the bottom of the essential spectrum of Σ (Theorems 3.4 and 3.6), and some compactness results of pmc surfaces (see Theorem 3.8 and Corollaries 3.9 and 3.12).

2. PRELIMINARIES

Let $M^n(c)$ be a space form, i.e., a simply connected n -dimensional manifold with constant sectional curvature c , and consider the product space $\bar{M} = M^n(c) \times \mathbb{R}$. Then the curvature tensor \bar{R} of \bar{M} is given by

$$(2.1) \quad \begin{aligned} \bar{R}(X, Y)Z = & c\{\langle Y, Z \rangle X - \langle X, Z \rangle Y - \langle Y, \xi \rangle \langle Z, \xi \rangle X + \langle X, \xi \rangle \langle Z, \xi \rangle Y \\ & + \langle X, Z \rangle \langle Y, \xi \rangle \xi - \langle Y, Z \rangle \langle X, \xi \rangle \xi\}, \end{aligned}$$

for any tangent vector fields X , Y and Z , where ξ is the unit vector field tangent to \mathbb{R} .

Now, let us consider Σ an isometrically immersed surface in $M^n(c) \times \mathbb{R}$. The second fundamental form σ of Σ is defined by the equation of Gauss

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y),$$

for any vector fields X and Y tangent to the surface, where $\bar{\nabla}$ and ∇ are the Levi-Civita connections on $M^n(c) \times \mathbb{R}$ and Σ , respectively. Then the mean curvature vector field \vec{H} of Σ is given by $\vec{H} = (1/2) \text{trace } \sigma$. The shape operator A and the normal connection ∇^\perp are defined by the equation of Weingarten

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V,$$

for any tangent vector field X and any normal vector field V .

We also have the Gauss equation of the surface Σ in $\bar{M} = M^n(c) \times \mathbb{R}$

$$(2.2) \quad \begin{aligned} \langle R(X, Y)Z, W \rangle = & \langle \bar{R}(X, Y)Z, W \rangle + \langle \sigma(Y, Z), \sigma(X, W) \rangle \\ & - \langle \sigma(X, Z), \sigma(Y, W) \rangle, \end{aligned}$$

where X, Y, Z and W are vector fields tangent to Σ and R is the curvature tensor corresponding to ∇ .

Definition 2.1. If the mean curvature vector field \vec{H} of a surface Σ is parallel in the normal bundle, i.e., $\nabla^\perp \vec{H} = 0$, then Σ is called a *pmc surface*. When $n = 2$, a pmc surface in $\mathbb{M}^2(c) \times \mathbb{R}$ is just a surface with constant mean curvature $H = |\vec{H}|$ and it is called a *cmc surface*.

In [2], the authors showed that the $(2, 0)$ -part of the quadratic form Q defined on a pmc surface Σ immersed in $\mathbb{M}^n(c) \times \mathbb{R}$ by

$$Q(X, Y) = 2\langle \sigma(X, Y), \vec{H} \rangle - c\langle X, \xi \rangle \langle Y, \xi \rangle$$

is holomorphic. When $n = 2$, this is just the Abresch-Rosenberg differential introduced in [1]. Then, in [3] and [12], the authors considered an operator S on a pmc surface Σ given by

$$2H\langle SX, Y \rangle = Q(X, Y) - \frac{\text{trace } Q}{2}\langle X, Y \rangle,$$

or, equivalently,

$$(2.3) \quad SX = \frac{1}{H}A_{\vec{H}}X - \frac{c}{2H}\langle X, T \rangle T + \frac{c}{4H}|T|^2X - HX,$$

where X and Y are vector fields tangent to Σ and T is the tangential component of the parallel vector field ξ . Two remarkable properties of this operator are the facts that S vanishes if and only if the $(2, 0)$ -part of Q vanishes and that S satisfies a Simons type equation.

Theorem 2.2 ([3], [12]). *Let Σ be an immersed non-minimal pmc surface in $\mathbb{M}^n(c) \times \mathbb{R}$. Then*

$$\frac{1}{2}\Delta|S|^2 = 2K_\Sigma|S|^2 + |\nabla S|^2,$$

where K_Σ is the Gaussian curvature of the surface.

Corollary 2.3. *Let Σ be an immersed non-minimal pmc surface in $\mathbb{M}^n(c) \times \mathbb{R}$ such that $\mu = \sup_\Sigma(|\sigma|^2 - (1/H)^2|A_{\vec{H}}|^2) < +\infty$, and let u be the function $u = |S|$. Then*

$$-\Delta u \leq au^3 + bu,$$

where a and b are constants depending on c , H and μ .

Proof. Let us consider the local orthonormal frame field $\{E_3 = \vec{H}/H, E_4, \dots, E_{n+1}\}$ in the normal bundle, and denote $A_\alpha = A_{E_\alpha}$.

From the definition (2.3) of S , we have, after a straightforward computation,

$$\det A_3 = H^2 - \frac{1}{2}|S|^2 - \frac{c^2}{16H^2}|T|^4 - \frac{c}{2H}\langle ST, T \rangle,$$

and then, using the Gauss equation (2.2) and the expression (2.1) of the curvature tensor \bar{R} of $\mathbb{M}^n(c) \times \mathbb{R}$, the Gaussian curvature K_Σ of our surface can be written as

$$K_\Sigma = c(1 - |T|^2) + H^2 - \frac{1}{2}|S|^2 - \frac{c^2}{16H^2}|T|^4 - \frac{c}{2H}\langle ST, T \rangle + \sum_{\alpha>3} \det A_\alpha.$$

Next, from Theorem 2.2, we obtain

$$(2.4) \quad \frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + 2\left(c(1 - |T|^2) + H^2 - \frac{1}{2}|S|^2 - \frac{c^2}{16H^2}|T|^4 - \frac{c}{2H}\langle ST, T \rangle + \sum_{\alpha>3} \det A_\alpha\right)|S|^2.$$

Since A_α is traceless for any $\alpha > 3$, we have

$$-2 \sum_{\alpha>3} \det A_\alpha = |\sigma|^2 - \frac{1}{H^2}|A_{\vec{H}}|^2 \leq \mu.$$

We note that $|\nabla|S|| \leq |\nabla S|$ and, since S is traceless, $|ST| = \frac{1}{\sqrt{2}}|T||S|$. Then, from (2.4), also using that the Schwarz inequality implies $|\langle ST, T \rangle| \leq |T||ST|$, it is easy to see that

$$\begin{aligned} \Delta|S| &\geq -|S|^3 + |S| \left(2c(1 - |T|^2) + 2H^2 - \frac{c^2}{8H^2} - \mu \right) - \frac{|c|}{\sqrt{2}H}|S|^2 \\ &\geq -|S|^3 + |S| \left(2\min\{c, 0\} + 2H^2 - \frac{c^2}{8H^2} - \mu \right) - \frac{|c|}{\sqrt{2}H}|S|^2 \\ &\geq -\left(1 + \frac{|c|}{2\sqrt{2}H}\right)|S|^3 - \left(\frac{|c|}{2\sqrt{2}H} - 2H^2 + \frac{c^2}{8H^2} - 2\min\{c, 0\} + \mu\right)|S| \end{aligned}$$

which completes the proof. \square

We end this section by recalling that a pmc surface Σ in $\mathbb{M}^n(c) \times \mathbb{R}$ satisfies a Sobolev inequality of the form

$$(2.5) \quad \forall f \in C_0^\infty(\Sigma), \quad \|f\|_2 \leq A_\Sigma \|\nabla f\|_1 + B_\Sigma \|f\|_1,$$

where $\|f\|_p = (\int_\Sigma |f|^p dv_\Sigma)^{1/p}$ is the L^p -norm of the function f and A_Σ and B_Σ are constants that depends only on the mean curvature H of the surface (see [16]).

3. THE MAIN RESULTS AND APPLICATIONS

Let Σ be an immersed surface in $\mathbb{M}^n(c) \times \mathbb{R}$ and $x_0 \in \Sigma$ a fixed a point. Consider the Riemannian distance function $d(x_0, x)$ to x_0 and the following open domains

$$B(R) = \{x \in \Sigma | d(x_0, x) < R\} \quad \text{and} \quad E(R) = \{x \in \Sigma | d(x_0, x) > R\}.$$

We can now state our main result.

Theorem 3.1. *Let Σ be a complete non-minimal pmc surface in $\mathbb{M}^n(c) \times \mathbb{R}$ such that the norm of its second fundamental form σ is bounded and*

$$(3.1) \quad \int_\Sigma |S|^2 dv_\Sigma < +\infty.$$

Then the function $u = |S|$ goes to zero uniformly at infinity. More precisely, there exist positive constants C_0 and C_1 depending on c , H and μ , where $\mu = \sup_\Sigma (|\sigma|^2 - (1/H)^2|A_{\vec{H}}|^2)$, and a positive radius R_Σ determined by the condition

$$C_1 \int_{E(R_\Sigma)} u^2 dv_\Sigma \leq 1 \text{ such that, for all } R \geq R_\Sigma,$$

$$\|u\|_{\infty, E(2R)} \leq C_0 \int_\Sigma u^2 dv_\Sigma.$$

Moreover, there exist some positive constants D_0 and E_0 depending on c , H and μ such that the inequality $\int_{\Sigma} u^2 dv_{\Sigma} \leq D_0$ implies

$$\|u\|_{\infty} \leq E_0 \int_{\Sigma} u^2 dv_{\Sigma}.$$

Proof. Since the function u satisfies the Sobolev inequality (2.5) and the inequality in Corollary 2.3, we can work as in the proof of [5, Theorem 4.1] to conclude. \square

Remark 3.2. When $n = 2$, we have $\mu = 0$ and it is easy to see that (3.1) implies that $|\sigma|$ is bounded, since a straightforward computation, using (2.3), shows that

$$|\sigma|^2 = \frac{1}{H^2} |A_{\vec{H}}|^2 = |S|^2 + \frac{c}{H} \langle ST, T \rangle + \frac{c^2 |T|^4}{8H^2} + 2H^2.$$

Corollary 3.3. Let Σ be a complete non-minimal cmc surface in $\mathbb{M}^2(c) \times \mathbb{R}$ with

$$\int_{\Sigma} |S|^2 dv_{\Sigma} < +\infty.$$

Then the function $u = |S|$ goes to zero uniformly at infinity. More precisely, there exist positive constants C_0 and C_1 depending on c and H , and a positive radius R_{Σ} determined by the condition $C_1 \int_{E(R_{\Sigma})} u^2 dv_{\Sigma} \leq 1$ such that, for all $R \geq R_{\Sigma}$,

$$\|u\|_{\infty, E(2R)} \leq C_0 \int_{\Sigma} u^2 dv_{\Sigma}.$$

Moreover, there exist some positive constants D_0 and E_0 depending on c and H such that the inequality $\int_{\Sigma} u^2 dv_{\Sigma} \leq D_0$ implies

$$\|u\|_{\infty} \leq E_0 \int_{\Sigma} u^2 dv_{\Sigma}.$$

In the following we will present some applications of Theorem 3.1. First we find a positive lower bound for the bottom of the essential spectrum of the Laplacian. For the sake of simplicity we will only consider the case when $c = \pm 1$.

Theorem 3.4. Let Σ be an immersed complete non-minimal cmc surface in $\mathbb{M}^2(c) \times \mathbb{R}$, $c = \pm 1$, with finite index $I_{\Sigma} < +\infty$ and

$$\int_{\Sigma} |S|^2 dv_{\Sigma} < +\infty.$$

Then we have

$$\lambda_{ess}^{\Delta} \geq \begin{cases} 2H^2, & \text{if } c = 1 \\ \frac{(4H^2 - 1)^2}{8H^2}, & \text{if } H > \frac{1}{2} \text{ and } c = -1. \end{cases}$$

Proof. Using equation (2.3), we can write the Jacobi operator of Σ as

$$J = \Delta + \left(|S|^2 + \frac{c}{H} \langle ST, T \rangle \right) + \frac{1}{8H^2} (c|T|^2 + 4H^2)^2.$$

We can see, from Corollary 3.3, that, for any $\varepsilon > 0$, there exists $R > 0$ such that $||S|^2 + (c/H) \langle ST, T \rangle| < \varepsilon$ in $\Sigma \setminus B(R)$. Thus, the index form associated to the

Jacobi operator J satisfies

$$I(f, f) \leq \int_{\Sigma} |\nabla f|^2 dv_{\Sigma} + \varepsilon \int_{\Sigma} f^2 dv_{\Sigma} - \frac{1}{8H^2} \int_{\Sigma} (c|T|^2 + 4H^2)^2 f^2 dv_{\Sigma},$$

for any $f \in C_0^{\infty}(\Sigma \setminus B(R))$.

Since the index is finite, we can choose $R > 0$ such that I is nonnegative in $\Sigma \setminus B(R)$ (see [14]). Hence we get

$$\int_{\Sigma} |\nabla f|^2 dv_{\Sigma} \geq \frac{1}{8H^2} \int_{\Sigma} ((c|T|^2 + 4H^2)^2 - \varepsilon) f^2 dv_{\Sigma}$$

for any $f \in C_0^{\infty}(\Sigma \setminus B(R))$.

Finally, we use the fact that $\lambda_{ess}^{\Delta} = \lim_{R \rightarrow \infty} \lambda_1(\Sigma \setminus B(R))$ to conclude the proof. \square

Corollary 3.5. *Let Σ be a surface in $\mathbb{M}^2(c) \times \mathbb{R}$ as in Theorem 3.4. Then there exist two positive numbers C and a such that*

$$\text{Vol}_{\Sigma}(B(R)) \geq Ce^{aR},$$

for any radius R big enough.

Proof. The conclusion follows from Theorem 3.4 and [4, Theorem 1]. \square

In the next theorem we have a result about the bottom of the essential spectrum of the Jacobi operator of Σ and a consequence on its Morse index.

Theorem 3.6. *Let Σ be a complete non-minimal cmc surface in $\mathbb{M}^2(c) \times \mathbb{R}$ with finite total curvature. Then we have*

$$\lambda_{ess}^J \leq \lambda_{ess}^{\Delta} - (2H^2 - 1).$$

Proof. From Corollary 3.3, we have that for any $\varepsilon > 0$, there exists $R > 0$ such that $||S|^2 + (c/H)\langle ST, T \rangle| < \varepsilon$ in $\Sigma \setminus B(R)$. Then the Jacobi operator J of the surface satisfies

$$\begin{aligned} J &= \Delta - |T|^2 + |S|^2 + \frac{c}{H} \langle ST, T \rangle + \frac{c^2 |T|^4}{8H^2} + 2H^2 \\ &\geq \Delta + 2H^2 - |T|^2 - \varepsilon \geq \Delta + 2H^2 - 1 - \varepsilon, \end{aligned}$$

which leads to the conclusion. \square

Corollary 3.7. *Let Σ be a complete cmc surface in $\mathbb{M}^2(c) \times \mathbb{R}$ with $H > \frac{1}{\sqrt{2}}$ and finite total curvature. Assume that*

$$\text{Vol}_{\Sigma}(B(R)) \leq Ce^{aR},$$

for any $R > 0$ and some positive constants C and $a < 2\sqrt{2H^2 - 1}$. Then Σ has infinite index.

Proof. We use Theorem 3.6 and [9, Theorem 3.1] to conclude that $\lambda_{ess}^J < 0$. \square

The next applications of our main result concern the compactness of pmc surfaces of finite total curvature.

Theorem 3.8. *Let Σ be a complete non-minimal pmc surface in $\mathbb{M}^n(c) \times \mathbb{R}$ with mean curvature vector field \vec{H} and such that the norm of its second fundamental form σ is bounded and*

$$\int_{\Sigma} |S|^2 dv_{\Sigma} < +\infty.$$

Then we have

- (1) *If $c > 0$ and $H^2 > (\mu + \sqrt{\mu^2 + c^2})/4$, then Σ is compact;*
- (2) *If $c < 0$ and $H^2 > (\mu - 2c + \sqrt{\mu^2 - 4c\mu + 5c^2})/4$, then Σ is compact,*

where $\mu = \sup_{\Sigma} (|\sigma|^2 - (1/H^2)|A_{\vec{H}}|^2)$.

Proof. As we have seen in the proof of Corollary 2.3, the Gaussian curvature K_{Σ} of Σ can be written as

$$K_{\Sigma} = c(1 - |T|^2) + H^2 - \frac{1}{2}|S|^2 - \frac{c^2}{16H^2}|T|^4 - \frac{c}{2H}\langle ST, T \rangle + \sum_{\alpha > 3} \det A_{\alpha},$$

where $A_{\alpha} = A_{E_{\alpha}}$, $\{E_3 = \vec{H}/H, E_4, \dots, E_{n+1}\}$ being a local orthonormal frame field in the normal bundle, and then

$$(3.2) \quad K_{\Sigma} \geq c(1 - |T|^2) + H^2 - \frac{1}{2}|S|^2 - \frac{c^2}{16H^2} - \frac{|c|}{2\sqrt{2}H}|S| + \sum_{\alpha > 3} \det A_{\alpha},$$

since $|\langle ST, T \rangle| \leq |T||ST|$ and $|ST| = (1/\sqrt{2})|T||S|$.

Next, if $c > 0$, since $2 \sum_{\alpha > 3} \det A_{\alpha} = -(|\sigma|^2 - (1/H^2)|A_{\vec{H}}|^2) \geq -\mu$, we get

$$K_{\Sigma} \geq -\frac{1}{2}|S|^2 - \frac{c}{2\sqrt{2}H}|S| + H^2 - \frac{c^2}{16H^2} - \frac{1}{2}\mu.$$

When $c < 0$, from (3.2), we have

$$K_{\Sigma} \geq c - \frac{1}{2}|S|^2 - \frac{c}{2\sqrt{2}H}|S| + H^2 - \frac{c^2}{16H^2} - \frac{1}{2}\mu.$$

In both cases, the hypotheses and Theorem 3.1, imply that the superior limit of K_{Σ} at infinity is positive. This means that the negative part K_{Σ}^{-} of K_{Σ} has compact support and, therefore, satisfies

$$\int_{\Sigma} K_{\Sigma}^{-} dv_{\Sigma} < +\infty.$$

It follows, from Huber's Theorem (see [19, Theorem 1]), that the positive part K_{Σ}^{+} of K_{Σ} also satisfies

$$\int_{\Sigma} K_{\Sigma}^{+} dv_{\Sigma} < +\infty.$$

Next, outside a compact set Ω we have $K_{\Sigma}^{+} \geq k > 0$, where

$$k = \begin{cases} \frac{16H^4 - 8(\mu - 2c)H^2 - c^2}{16H^2}, & \text{if } c > 0 \\ \frac{16H^4 - 8\mu H^2 - c^2}{16H^2}, & \text{if } c < 0, \end{cases}$$

and then $\text{Vol}(\Sigma \setminus \Omega) < +\infty$. Since the volume of a complete non-compact surface is infinite (see [15]), it follows that our surface Σ is compact. \square

When $n = 2$, we use Remark 3.2 to obtain the following result.

Corollary 3.9. *Let Σ be a complete cmc surface in $\mathbb{M}^2(c) \times \mathbb{R}$ such that*

$$\int_{\Sigma} |S|^2 dv_{\Sigma} < +\infty.$$

Then we have

- (1) *If $c > 0$ and $H > \sqrt{c}/2$, then Σ is compact;*
- (2) *If $c < 0$ and $H > (\sqrt{\sqrt{5} + 2}/2)\sqrt{-c} \approx 1,02\sqrt{-c}$, then Σ is compact.*

In the following, we will need a result obtained in [12].

Theorem 3.10 ([13]). *Let Σ be a pmc surface in $\mathbb{M}^n(c) \times \mathbb{R}$. Then we have*

$$\frac{1}{2}\Delta|T|^2 = |A_N|^2 + c|T|^2(1 - |T|^2) - \sum_{\alpha=3}^{n+1} |A_{\alpha}T|^2,$$

where N is the normal part of the vector field ξ , $\{E_3, \dots, E_{n+1}\}$ is a local orthonormal frame field in the normal bundle and $A_{\alpha} = A_{E_{\alpha}}$.

Theorem 3.11. *Let Σ be a complete non-minimal pmc surface in $\mathbb{M}^n(c) \times \mathbb{R}$, $c < 0$, with mean curvature vector field \vec{H} and such that the norm of its second fundamental form σ is bounded and*

$$\int_{\Sigma} (|S|^2 + |N|^2) dv_{\Sigma} < +\infty.$$

If $H^2 > (\mu + \sqrt{\mu^2 + c^2})/4$, where $\mu = \sup_{\Sigma} (|\sigma|^2 - (1/H^2)|A_{\vec{H}}|^2)$, then Σ is compact.

Proof. From Theorem 3.10, as $|N|^2 = 1 - |T|^2$, we have

$$(3.3) \quad -\frac{1}{2}\Delta|N|^2 = |A_N|^2 + c|N|^2(1 - |N|^2) - \sum_{\alpha=3}^{n+1} |A_{\alpha}T|^2.$$

Next, since $\nabla \xi = 0$ implies $\nabla_X^{\perp} N = -\sigma(X, T)$, one obtains

$$2|N|X(|N|) = X(|N|^2) = 2\langle \nabla_X^{\perp} N, N \rangle = -2\langle A_N T, X \rangle,$$

and then

$$(3.4) \quad |N|^2 |\nabla |N||^2 = |A_N T|^2.$$

Replacing (3.4) in (3.3), we get that

$$-|N|^3 \Delta|N| = (|A_N|^2 + c|N|^2(1 - |N|^2))|N|^2 - \sum_{\alpha=3}^{n+1} |A_{\alpha}T|^2 |N|^2 + |A_N T|^2,$$

which gives

$$-|N|^3 \Delta|N| \leq (|\sigma|^2 + c(1 - |N|^2))|N|^4,$$

since, using the Schwarz inequality, we can see that $\sum_{\alpha=3}^{n+1} |A_{\alpha}T|^2 |N|^2 \geq |A_N T|^2$. It follows that there exists a constant d such that

$$-\Delta|N| \leq -c|N|^3 + d|N|.$$

Since the function $w = |N|$ also satisfies the Sobolev inequality (2.5) and, by hypothesis,

$$\int_{\Sigma} w^2 dv_{\Sigma} \leq +\infty,$$

we can again work as in the proof of [5, Theorem 4.1] to prove that also w goes to zero uniformly at infinity.

Now, from (3.2), we get

$$K_{\Sigma} \geq c|N|^2 + H^2 - \frac{1}{2}|S|^2 - \frac{c^2}{16H^2} - \frac{|c|}{2\sqrt{2}H}|S| - \frac{1}{2}\mu,$$

which, together with Theorem 3.1, shows that the superior limit of K_{Σ} is positive. This means that we can use the same arguments as in the proof of Theorem 3.8 to conclude. \square

Again using Remark 3.2 we have the following corollary.

Corollary 3.12. *Let Σ be a complete non-minimal cmc surface in $\mathbb{M}^2(c) \times \mathbb{R}$, $c < 0$, such that*

$$H > \frac{\sqrt{-c}}{2} \quad \text{and} \quad \int_{\Sigma} (|S|^2 + |N|^2) dv_{\Sigma} < +\infty,$$

Then Σ is compact.

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